How Conformal a Sensing Matrix Satisfying the Restricted Isometry Property Can Be: Compressive Sensing Meets Plane Geometry

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Outline

- Background and Motivation
- Problem Statement and Contribution
- Problem Formulation
- Main Results
- Application in Compressed-Domain Interference Cancellation
- Conclusions

Background and Motivation

- CS (compressive sensing) exploits the sparsity inherent in real-world signals so as to facilitate efficient data acquisition, storage, and processing
  - Far less samples than that required by Nyquist sampling theorem suffice to capture all information of sparse signals
- A CS system is typically described by an underdetermined linear equation

$$\mathbf{y} = \mathbf{\Phi x} + \mathbf{w}$$

  - $\mathbf{x} \in \mathbb{R}^p$: sparse signal vector with K nonzero entries $(K \ll p)$
  - $\mathbf{y} \in \mathbb{R}^m$: measurement vector
  - $\mathbf{\Phi} \in \mathbb{R}^{m \times p}$: sensing matrix $(m < p)$
  - $\mathbf{w} \in \mathbb{R}^m$: noise vector

Background and Motivation

- Successful signal recovery relies on the restricted isometry property (RIP) of the sensing matrix $\mathbf{\Phi}$

There exists $0 < \delta < 1$ such that, for all K-sparse $\mathbf{u}$ and $\mathbf{v}$, we have

$$(1-\delta)\|\mathbf{u}-\mathbf{v}\|_2^2 \le \left\|\mathbf{\Phi}(\mathbf{u}-\mathbf{v})\right\|_2^2 \le (1+\delta)\|\mathbf{u}-\mathbf{v}\|_2^2$$

In particular,

$$(1-\delta)\|\mathbf{u}\|_2^2 \le \left\|\mathbf{\Phi}\mathbf{u}\right\|_2^2 \le (1+\delta)\|\mathbf{u}\|_2^2$$

- If $\mathbf{\Phi}$ satisfies RIP with a small restricted isometry constant (RIC) δ
  - $\left\|\mathbf{\Phi}(\mathbf{u}-\mathbf{v})\right\|_2^2 \approx \|\mathbf{u}-\mathbf{v}\|_2^2$
  - $\left\|\mathbf{\Phi}\mathbf{u}\right\|_2^2 \approx \|\mathbf{u}\|_2^2$
  - Distance/norm largely preserved in the compressed domain
    - ✓ Robustness against noise perturbation

Background and Motivation

- RIP characterizes signal recoverability in terms of distance/norm
- An alternative metric for robustness of signal reconstruction is the angle between two compressed sparse vectors $\alpha = \measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v})$
  - If $\measuredangle(\mathbf{u}, \mathbf{v}) = \theta$, $\measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v}) \approx \theta$ is preferred
- In the literature of CS, the angle between two compressed sparse vectors $\mathbf{\Phi u}$ and $\mathbf{\Phi v}$ plays an important role in many studies regarding stability analyses and performance evaluations

Background and Motivation

- For example, the upper bound of $|\cos(\alpha)|$, $\alpha = \measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v})$ is central to characterization of the achievable system performance in
  - Compressive domain interference cancellation[1]
  - RIP-based analysis of the orthogonal matching pursuit (OMP) algorithm[2]
  - Study of democratic nature of random sensing matrices[3]
  - Parameter estimation base on compressed measurements[4]
  - Detection of sparse signals[5]

[1] M. A. Davenport, P .T. Boufounos, and R. G. Baraniuk, "Compressive domain interference cancellation," *Proc. Workshop on Signal Processing with Adaptive Sparse Structured Representation (SPARS),* Saint-Malo, France, 2009.

[2] M. A. Davenport and M. B. Wakin, "Analysis of orthogonal matching pursuit using the restricted isometry property," *IEEE Trans. Information Theory*, vol. 56, no. 9, pp. 4395-4401, Sept. 2010.

[3] M. A. Davenport, J. N. Laska, P .T. Boufounos, and R. G. Baraniuk, "A simple proof that random matrix are democratic," technical report, Rice University, USA, Nov. 2009.

[4] M. A. Davenport, P .T. Boufounos, M. B. Wakin, and R. G. Baraniuk, "Signal Processing with compressive measurements," *IEEE J. Selected Topics in Signal Processing*, vol. 4, no. 2, pp, 445-460, April 2010.

[5] J. Haupt and R. Nowark, "Compressive sampling for signal detection," *Proc. ICASSP 2007*, vol. 3, pp. 1509-1512.

Background and Motivation

- Existing approach to deriving an upper bound for $|\cos(\measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v}))|$ under RIP:
  - Recall

$$\cos(\measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v})) = \frac{\langle \mathbf{\Phi u}, \mathbf{\Phi v} \rangle}{\|\mathbf{\Phi u}\|_2 \|\mathbf{\Phi v}\|_2} \overset{(a)}{=} \frac{\{\|\mathbf{\Phi u} + \mathbf{\Phi v}\|_2^2 - \|\mathbf{\Phi u} - \mathbf{\Phi v}\|_2^2\}}{4\|\mathbf{\Phi u}\|_2 \|\mathbf{\Phi v}\|_2}$$

  - (a) holds by using polarization identity
  - Use RIP to bound $\|\mathbf{\Phi u}\|_2$, $\|\mathbf{\Phi v}\|_2$, and $\|\mathbf{\Phi u} \pm \mathbf{\Phi v}\|_2$, leading to

$$|\cos\measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v})| \le \min\left\{\frac{\delta + 2|\cos\theta|}{2(1-\delta)},\ 1\right\} \quad (1)$$

- "Algebraic" approach known to yield the worst-case estimate[6]

[6] M. E. Davies and R. Gribonval, "Restricted isometry constants where l^p sparse recovery can fail for $0 < p \le 1$," *IEEE Trans. Information Theory*, vol. 55, no. 5, pp. 2203-2214, May 2009.

Background and Motivation

- Is the upper bound (1) the entire story under RIP? Any other ways out to do better?
- We try to be even more aggressive, and ask the following question:

Is there any hope of explicitly specifying the achievable $\alpha = \measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v})$ under the RIP constraint on the sensing matrix $\mathbf{\Phi}$?

Problem Statement and Contribution

- In this work, we thus pose and study the fundamental problem:

Given: Two K-sparse vectors $\mathbf{u}$ and $\mathbf{v}$ with $\measuredangle(\mathbf{u}, \mathbf{v}) = \theta$

Assumption: Sensing matrix $\mathbf{\Phi}$ satisfies RIP with RIC δ

Goal: Identify the maximal and minimal achievable $\alpha = \measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v})$, denoted by $\alpha_{\max}$ and $\alpha_{\min}$

  - To simplify notation, the dependence of $\alpha_{\max}$ and $\alpha_{\min}$ on θ is omitted in the sequel

Problem Statement and Contribution

- Specific technical contributions
  - By exploiting **geometric interpretations of the RIP** and the well-known **law of cosines**, we propose a **plane geometry** based problem formulation
  - All the feasible compressed vector pairs under RIP can be tractably depicted via certain auxiliary triangles in a simple diagram on the 2-D plane
    - Suffice to seek for two triangles whose top vertexes yield, respectively, the maximal and minimal angles
  - By conducting **plane geometry analysis** with respect to the constructed diagram, **closed-form formulae** for $\alpha_{\max}$ and $\alpha_{\min}$ can be obtained
  - Improved estimates of achievable angles as compared with the solutions obtained using algebraic approach

Problem Statement and Contribution

- Significance and Impacts
  - Original study of the conformal (angle-preserving) characteristic of compressive maps in the literature
  - Contribute to fundamental understanding of the geometry of compressed (measurement) space
  - Facilitate more accurate performance evaluations for many CS algorithms and systems

Problem Formulation

- Underlying assumptions
  - Since the angle between two vectors is invariant with respect to scaling, we assume without loss of generality that $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1$
  - Assume $0 \le \measuredangle(\mathbf{u}, \mathbf{v}) \le \pi / 2$ without loss of generality
    - If $\measuredangle(\mathbf{u}, \mathbf{v}) > \pi / 2$, we can then consider $\measuredangle(\mathbf{u}, -\mathbf{v}) = \pi - \measuredangle(\mathbf{u}, \mathbf{v}) < \pi / 2$ instead

Problem Formulation

- All we know about $\mathbf{\Phi u}$ and $\mathbf{\Phi v}$ is the norm/distance constraints imposed by RIP
- To identify $\alpha_{\max}$ and $\alpha_{\min}$, we need an explicit connection between $\alpha = \measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v})$ and norm/distance
  - Hopefully, a formula that can provide distinctive insights from a geometric perspective

Solution: the law of cosines

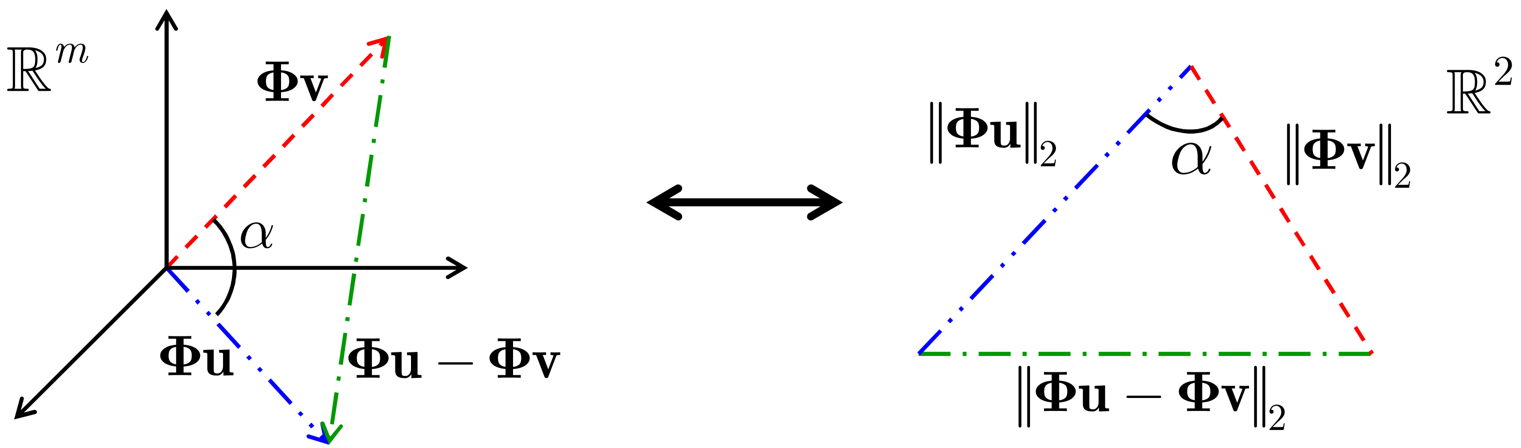
$$\cos\alpha = \frac{\|\mathbf{\Phi u}\|_2^2 + \|\mathbf{\Phi v}\|_2^2 - \|\mathbf{\Phi}(\mathbf{u}-\mathbf{v})\|_2^2}{2\|\mathbf{\Phi u}\|_2 \|\mathbf{\Phi v}\|_2} \qquad (2)$$

Problem Formulation

- Major advantage: Provide a simple and concrete geometric view of angle between $\mathbf{\Phi u}$ and $\mathbf{\Phi v}$

✓ $\|\mathbf{\Phi u}\|_2$, $\|\mathbf{\Phi v}\|_2$ and $\|\mathbf{\Phi u} - \mathbf{\Phi v}\|_2$ as the three sides of a triangle on the two-dimensional plane

✓ $\alpha = \measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v})$ is computed as the angle determined by one vertex of the triangle



Problem Formulation

- Recall the law of cosines

$$\cos\alpha = \left(\frac{\|\mathbf{\Phi u}\|_2^2 + \|\mathbf{\Phi v}\|_2^2 - \|\mathbf{\Phi u} - \mathbf{\Phi v}\|_2^2}{2\|\mathbf{\Phi u}\|_2\|\mathbf{\Phi v}\|_2}\right) = -\left(\frac{\|\mathbf{\Phi u}\|_2^2 + \|\mathbf{\Phi v}\|_2^2 - \|\mathbf{\Phi u} + \mathbf{\Phi v}\|_2^2}{2\|\mathbf{\Phi u}\|_2\|\mathbf{\Phi v}\|_2}\right)$$

- Need all information about $\|\mathbf{\Phi u}\|_2^2$, $\|\mathbf{\Phi v}\|_2^2$, $\|\mathbf{\Phi u} - \mathbf{\Phi v}\|_2^2$ and $\|\mathbf{\Phi u} + \mathbf{\Phi v}\|_2^2$ to characterize α

Under RIP,

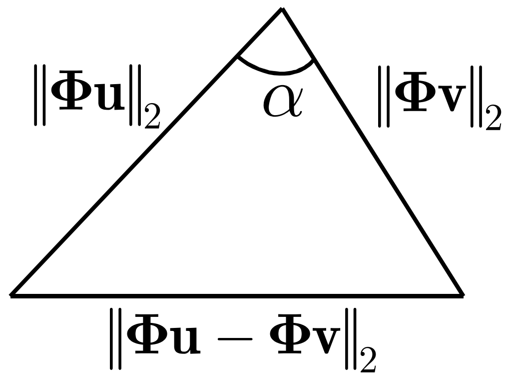
(i) $1-\delta \le \|\mathbf{\Phi u}\|_2^2 \le 1+\delta,\ 1-\delta \le \|\mathbf{\Phi v}\|_2^2 \le 1+\delta$

(ii) $\underbrace{2(1-\delta)(1-\cos\theta)}_{\triangleq d_{\min}^2} \le \|\mathbf{\Phi u} - \mathbf{\Phi v}\|_2^2 \le \underbrace{2(1+\delta)(1-\cos\theta)}_{\triangleq d_{\max}^2}$

(iii) $\underbrace{2(1-\delta)(1+\cos\theta)}_{\triangleq \tilde{d}_{\min}^2} \le \|\mathbf{\Phi u} + \mathbf{\Phi v}\|_2^2 \le \underbrace{2(1+\delta)(1+\cos\theta)}_{\triangleq \tilde{d}_{\max}^2}$

Problem Formulation

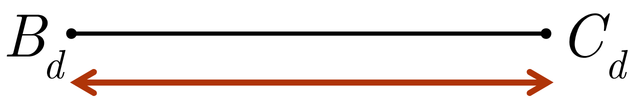
- Question: How to translate the "algebraic " inequalities (i)~(iii) into concrete "geometric" depictions?
- Main ideas behind:
  - Geometric depiction of law of cosines:



- Pick a feasible distance $\|\Phi(\mathbf{u}-\mathbf{v})\|_2 = d,\ \ d_{\min} \le d \le d_{\max}$ under constraint (ii) as the bottom of a triangle
- Then determine the top vertex with which
  - ✓ Length of two sides fulfill constraint (i)
  - ✓ The resultant $\|\Phi(\mathbf{u}+\mathbf{v})\|_2$ satisfies constraint (iii)

Problem Formulation

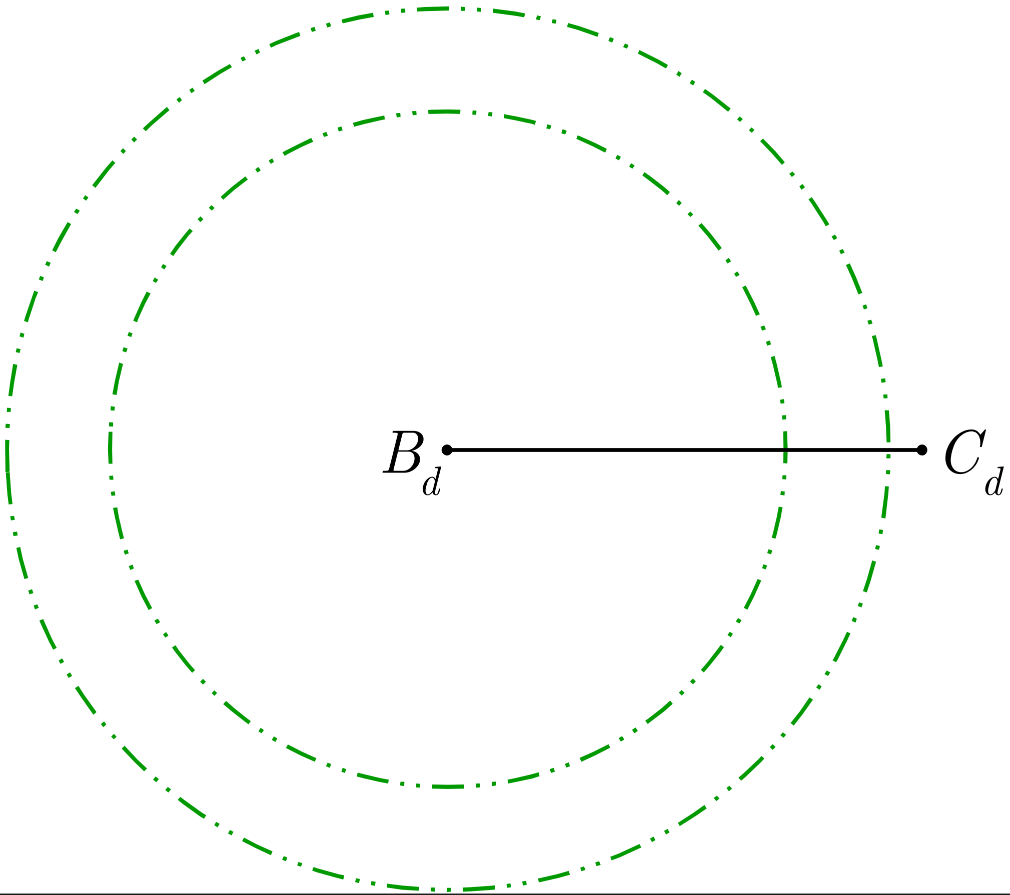
- Pick a plausible compressed distance $\|\mathbf{\Phi u} - \mathbf{\Phi v}\|_2 = d$, which fulfills constraint (ii), namely, $d_{\min} \le d \le d_{\max}$ and draw a line segment $\overline{B_d C_d}$ on the plane with $|\overline{B_d C_d}| = d$



$$|\overline{B_d C_d}| = \|\mathbf{\Phi u} - \mathbf{\Phi v}\|_2 = d$$

Problem Formulation

- Determine the top vertex set $\hat{\mathbf{\Omega}}(d)$ s.t. if $D_d \in \hat{\mathbf{\Omega}}(d)$, then $|\overline{D_d B_d}| = \|\mathbf{\Phi u}\|_2$ and $|\overline{D_d C_d}| = \|\mathbf{\Phi v}\|_2$ satisfy the norm constraint (i) $1-\delta \le \|\mathbf{\Phi u}\|_2^2 \le 1+\delta,\ 1-\delta \le \|\mathbf{\Phi v}\|_2^2 \le 1+\delta$



Problem Formulation

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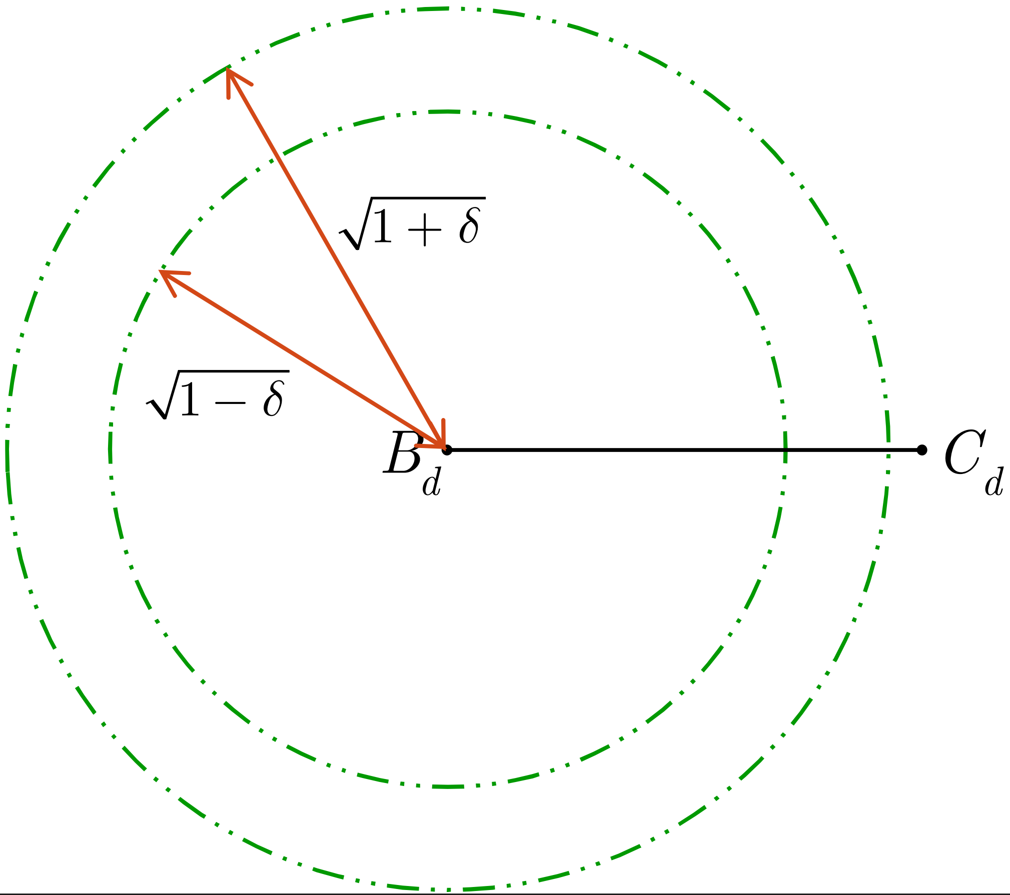
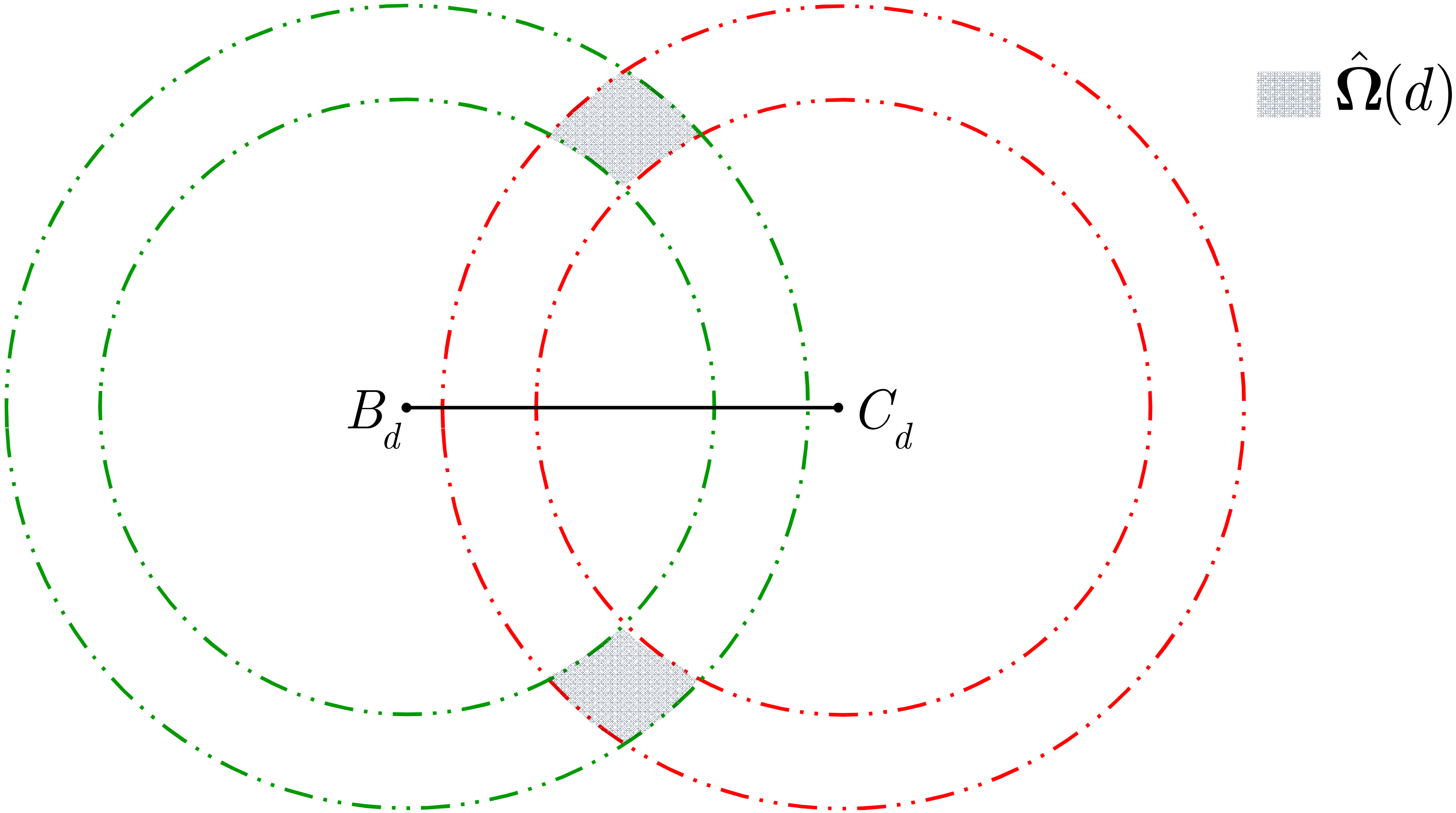


Problem Formulation

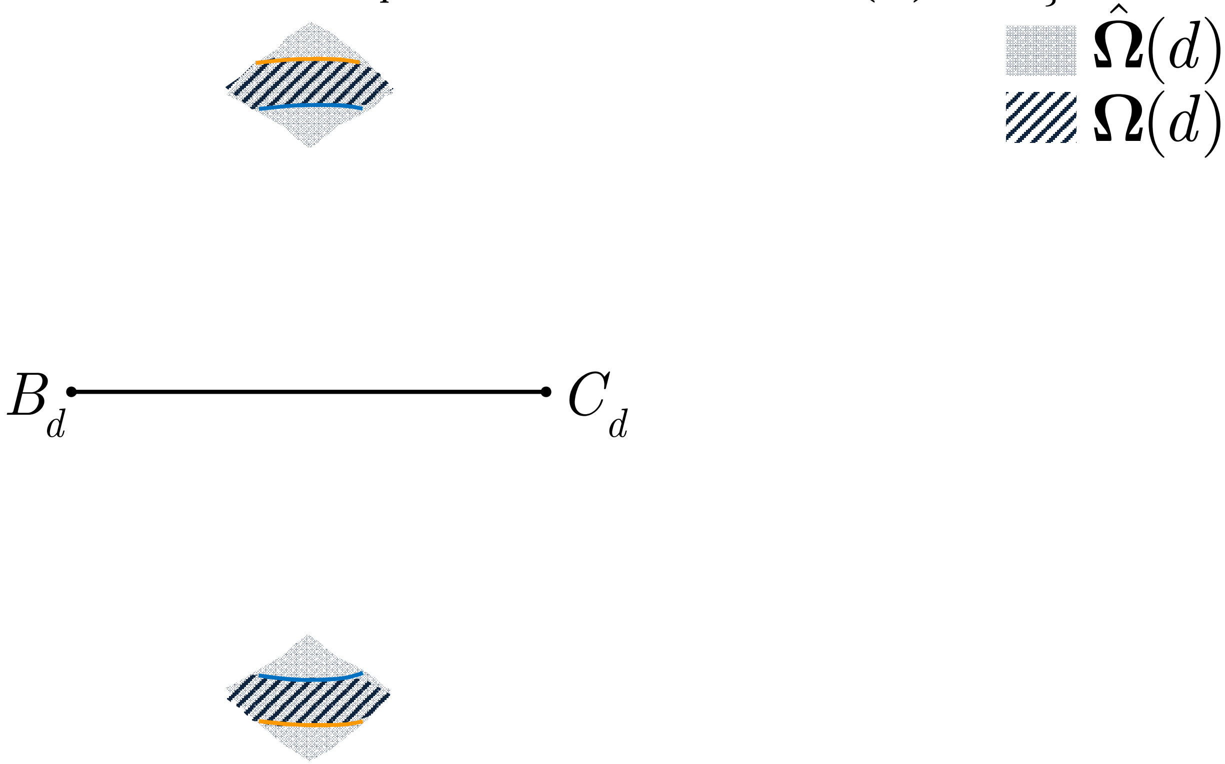
- Determine the top vertex set $\hat{\mathbf{\Omega}}(d)$ s.t. if $D_d \in \hat{\mathbf{\Omega}}(d)$, then $|\overline{D_d B_d}| = \|\mathbf{\Phi u}\|_2$ and $|\overline{D_d C_d}| = \|\mathbf{\Phi v}\|_2$ satisfy the norm constraint (i) $1-\delta \le \|\mathbf{\Phi u}\|_2^2 \le 1+\delta,\ 1-\delta \le \|\mathbf{\Phi v}\|_2^2 \le 1+\delta$



Problem Formulation

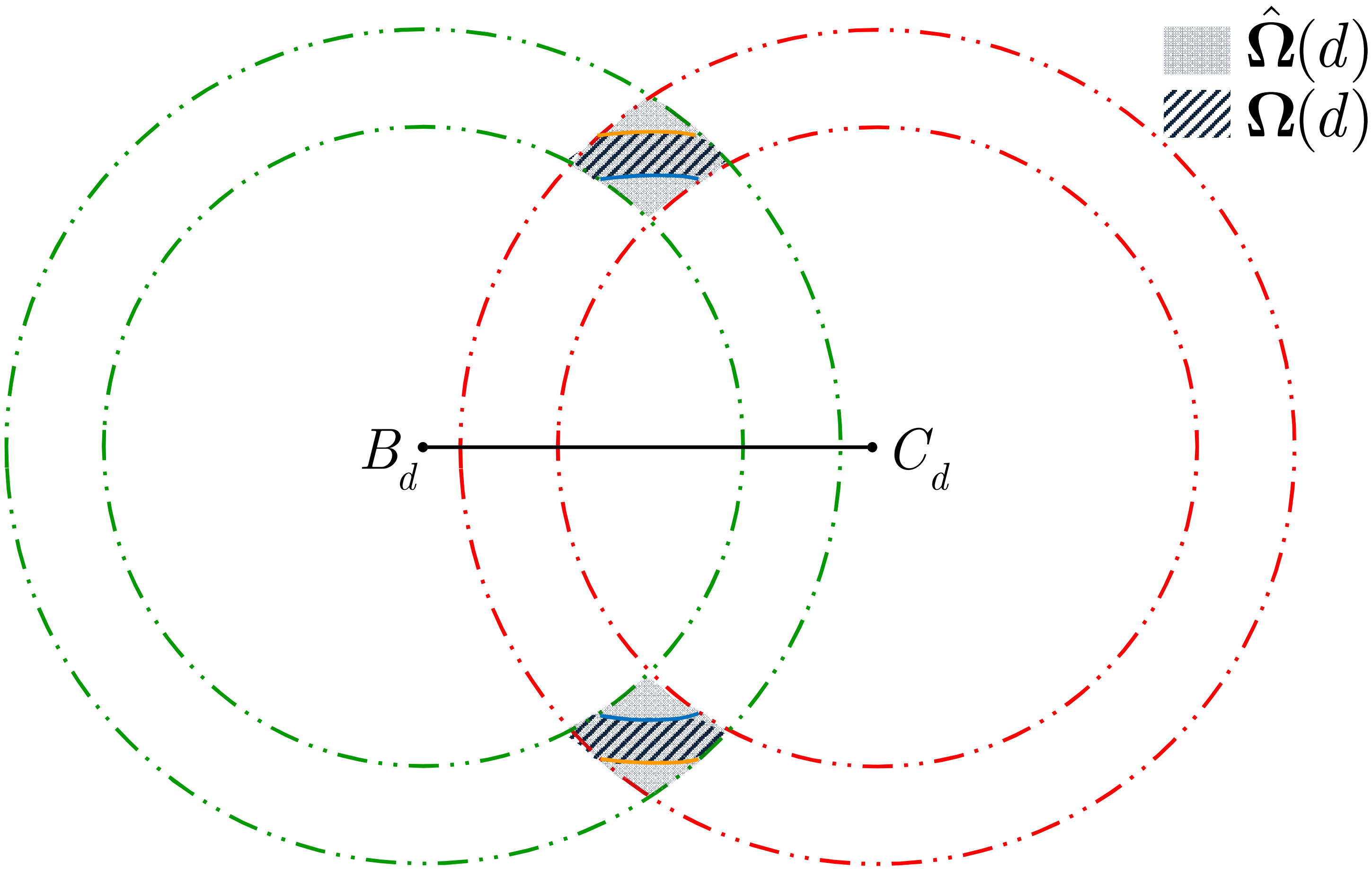
- Based on $\hat{\boldsymbol{\Omega}}(d)$, determine those $D_d \in \hat{\boldsymbol{\Omega}}(d)$ that fulfill constraint (iii), namely,

$$\boldsymbol{\Omega}(d) \triangleq \left\{ D_d \mid D_d \in \hat{\boldsymbol{\Omega}}(d), \underbrace{\frac{\tilde{d}_{\min}^2 + d^2}{2} \leq \left| \overline{D_d B_d} \right|^2 + \left| \overline{D_d C_d} \right|^2 \leq \frac{\tilde{d}_{\max}^2 + d^2}{2}}_{\text{an equivalent form of constraint (iii)}} \right\}$$



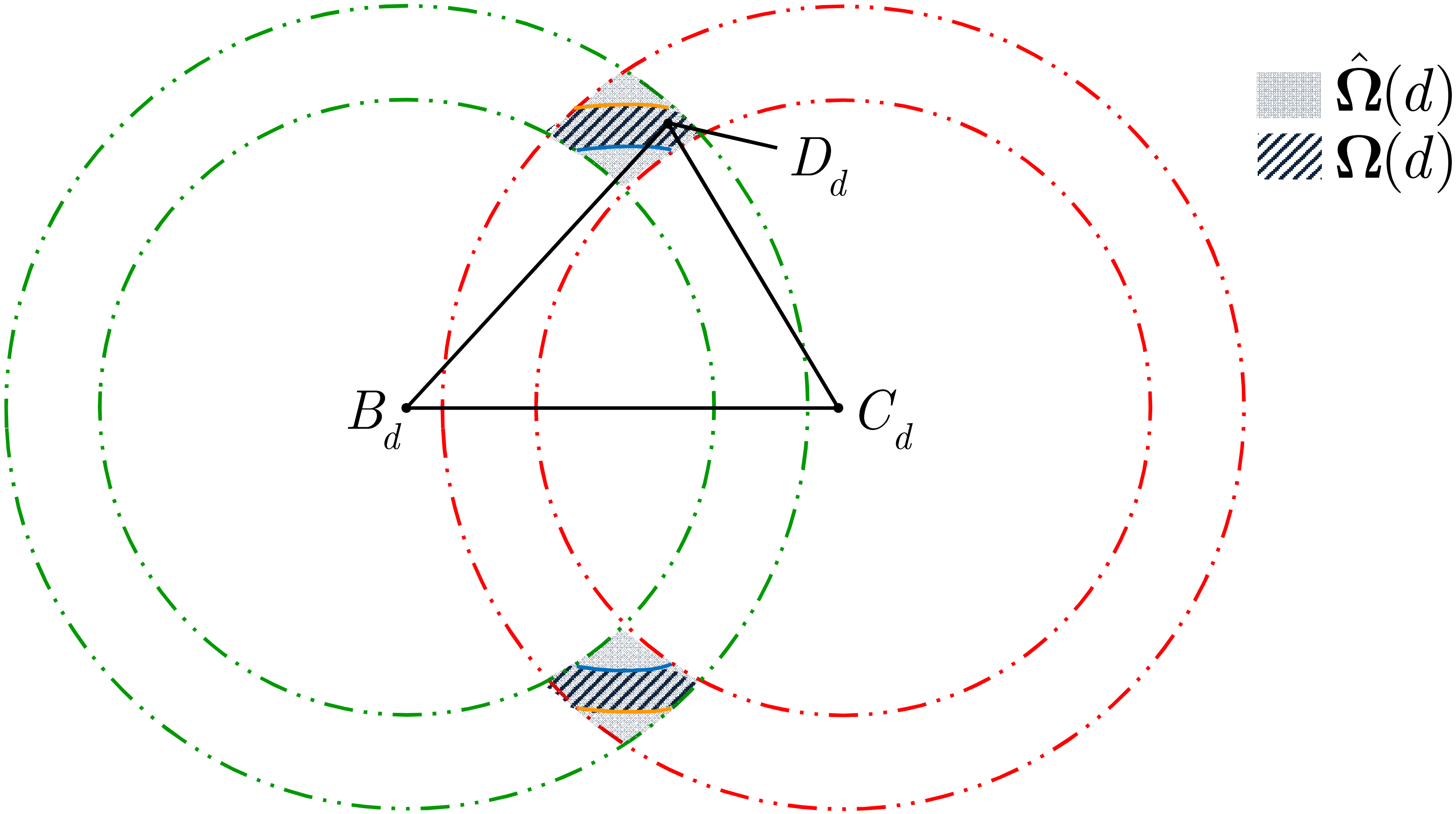
Problem Formulation

- For a fixed $\|\mathbf{\Phi u} - \mathbf{\Phi v}\|_2 = d,\ d_{\min} \le d \le d_{\max}$, there is a corresponding ***feasible top vertex set*** $\mathbf{\Omega}(d)$



Problem Formulation

- For a fixed $D_d \in \mathbf{\Omega}(d)$, $\Delta D_d B_d C_d$ is with three sides $(|\overline{D_d B_d}|,|\overline{D_d C_d}|,|\overline{B_d C_d}|) = (\|\mathbf{\Phi u}\|_2, \|\mathbf{\Phi v}\|_2, \|\mathbf{\Phi(u-v)}\|_2 = d)$
- $\alpha = \measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v}) = \measuredangle B_d D_d C_d$



Problem Formulation

- With $\mathbf{\Omega}(d)$ constructed as above, what can we do next?
  - May first find $\alpha_{\max}(d)$ and $\alpha_{\min}(d)$ for each d:

$$\alpha_{\max}(d) = \max_{D_d \in \mathbf{\Omega}(d)} \measuredangle B_d D_d C_d, \ \alpha_{\min}(d) = \min_{D_d \in \mathbf{\Omega}(d)} \measuredangle B_d D_d C_d$$

  - Then determine the global $\alpha_{\max}$ and $\alpha_{\min}$ as

$$\alpha_{\max} = \max_{d_{\min} \le d \le d_{\max}} \alpha_{\max}(d), \ \alpha_{\min} = \min_{d_{\min} \le d \le d_{\max}} \alpha_{\min}(d)$$

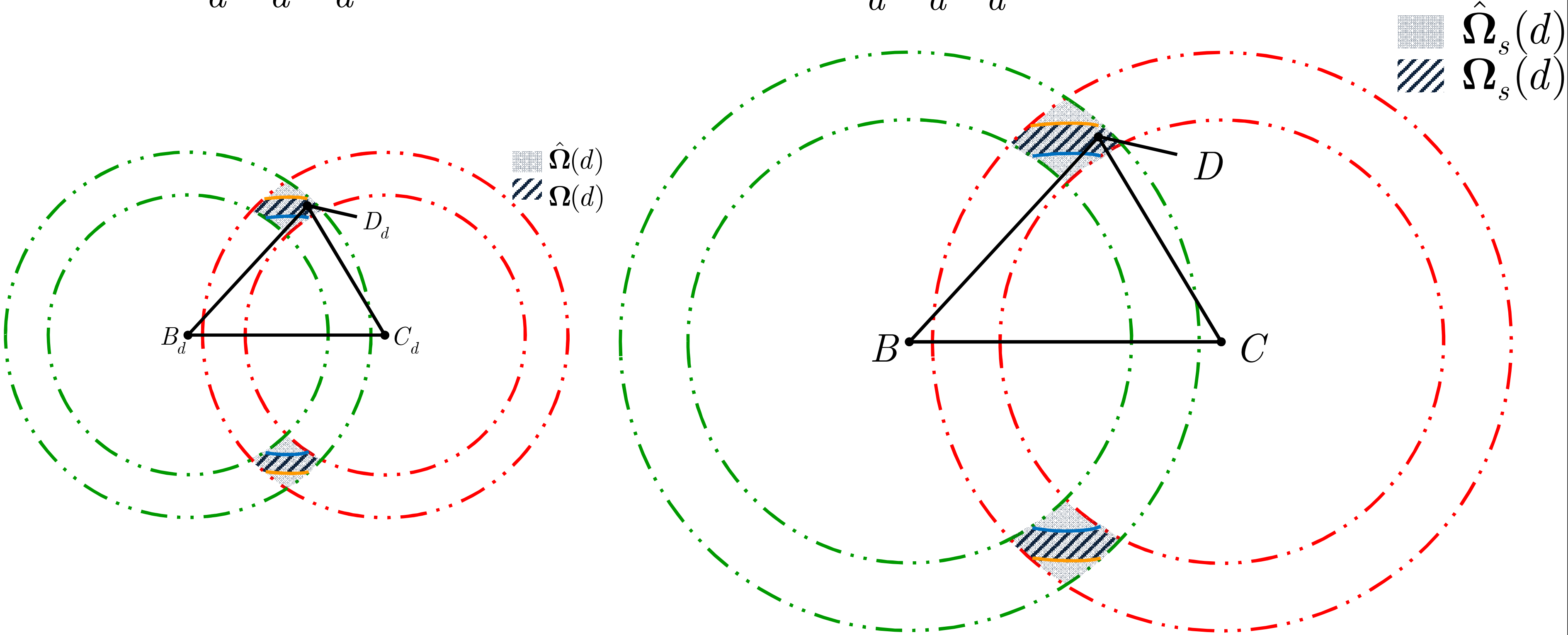
- A daunting task since there are infinitely many plausible d!

Question: Is it possible to obtain *a single diagram* which *jointly* depicts all plausible triangles $\{\triangle B_d D_d C_d \mid D_d \in \mathbf{\Omega}(d), \ d_{\min} \le d \le d_{\max}\}$ so as to facilitate a tractable and unified analysis?

Solution: Resort to "similarity" technique in plane geometry analysis!

Problem Formulation

- Similarity → Conformal
- $\Delta D_d B_d C_d \sim \Delta DBC$, thus $\measuredangle B_d D_d C_d = \measuredangle BDC$



By constructing a similar dilated diagram with an appropriate scale, there is a simple way of obtaining a ***joint similar feasible top vertex set***!

Problem Formulation

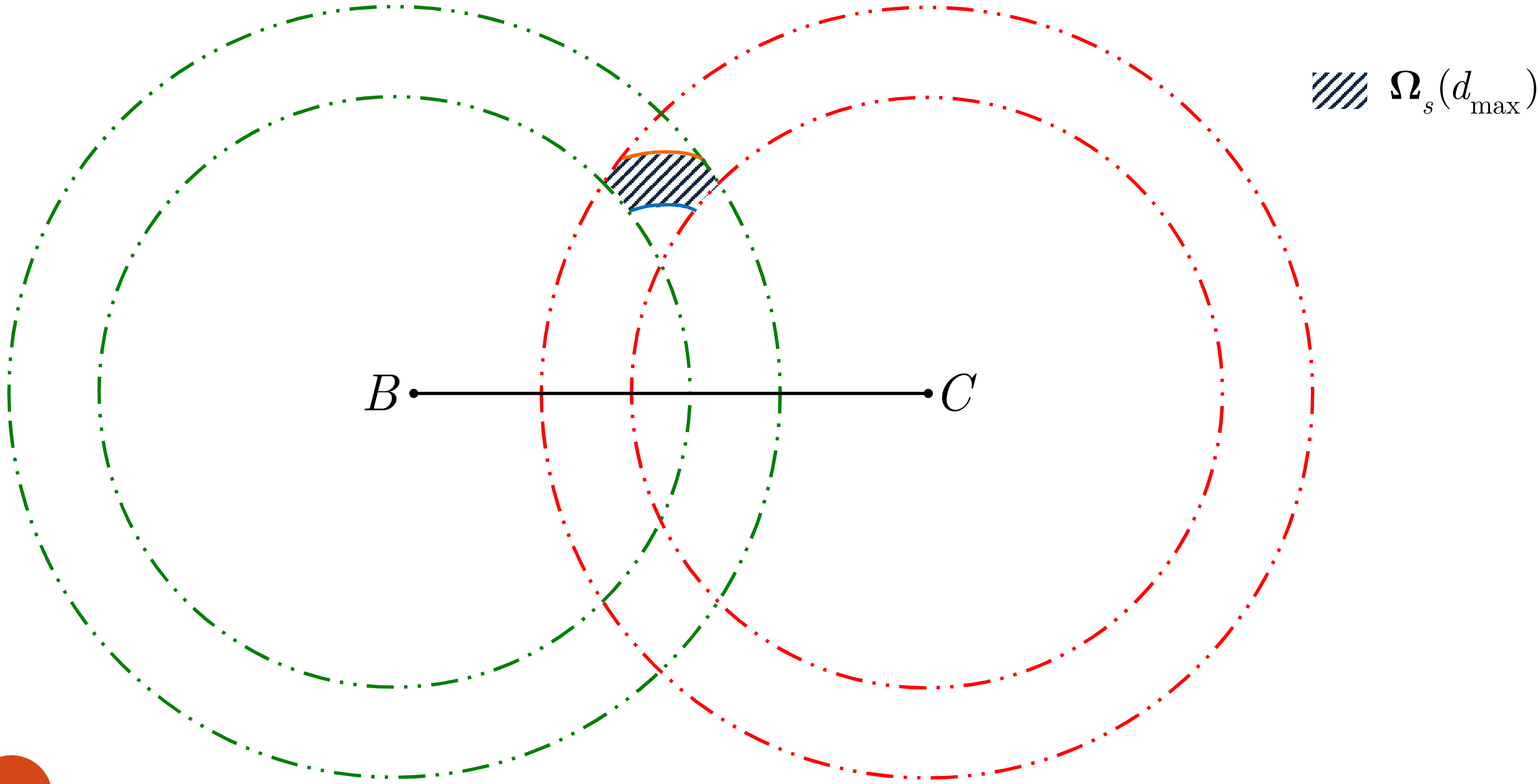
- Proposed approach to constructing the diagram for the ***joint similar feasible top vertex set***, denoted by $\boldsymbol{\Omega}$
  - For each plausible compressed distance $\|\boldsymbol{\Phi}(\mathbf{u}-\mathbf{v})\|_2 = d$, uniformly stretch all the objects in the diagram by the scale $d_{\max}/d\ (\geq 1)$ to obtain a dilated similar diagram $\boldsymbol{\Omega}_s(d)$
    - All dilated diagrams are with a common enlarged compressed distance

$$\|\boldsymbol{\Phi}(\mathbf{u}-\mathbf{v})\|_2 \times \frac{d_{\max}}{d} = d_{\max}$$

  - With $|\overline{BC}| = d_{\max}$ as a common baseline, put all the dilated similar diagrams $\boldsymbol{\Omega}_s(d)$'s one on top of another to obtain the diagram for $\boldsymbol{\Omega}$

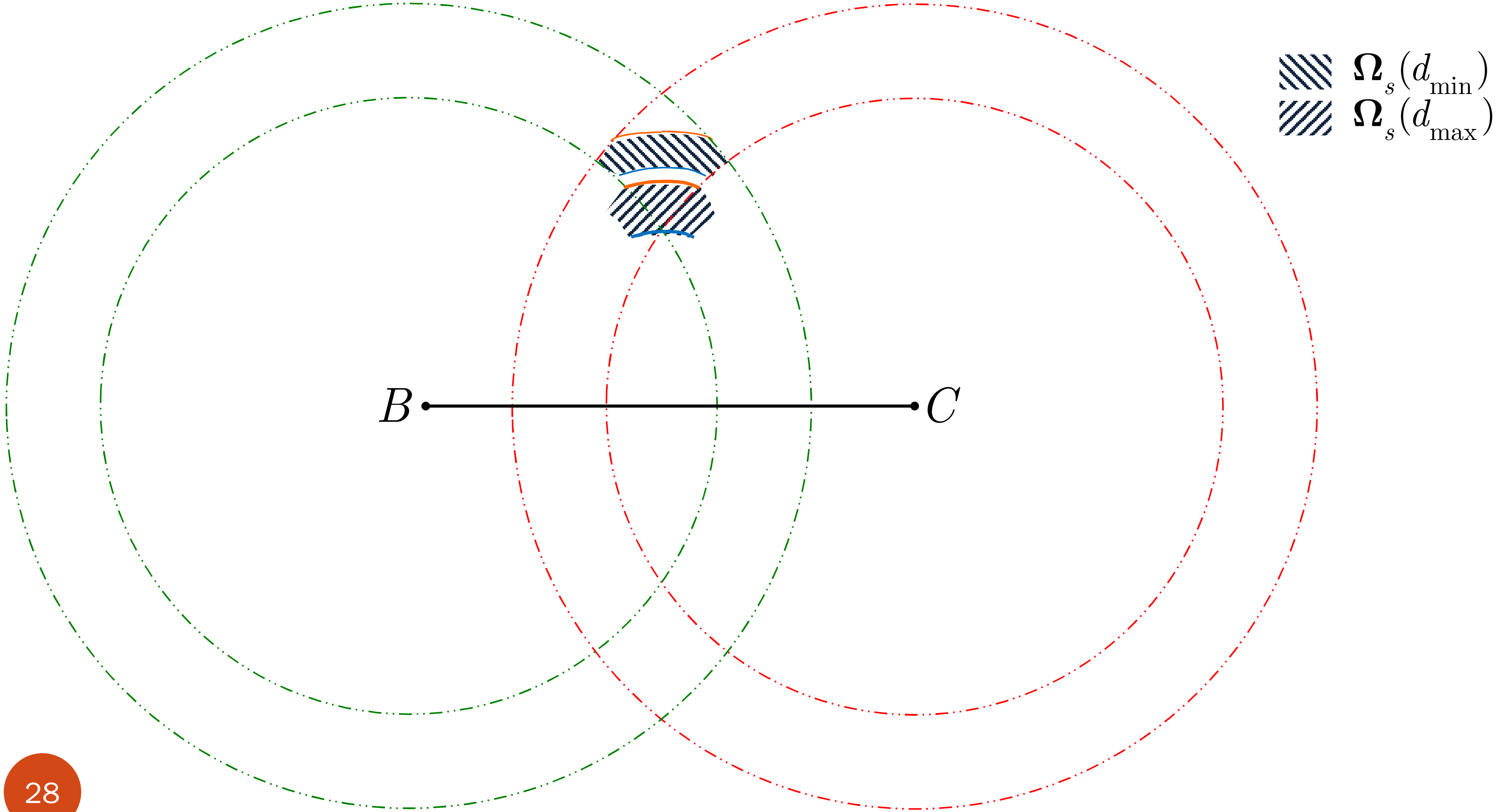
Problem Formulation

- Construction of the *joint similar feasible top vertex set* $\mathbf{\Omega}$



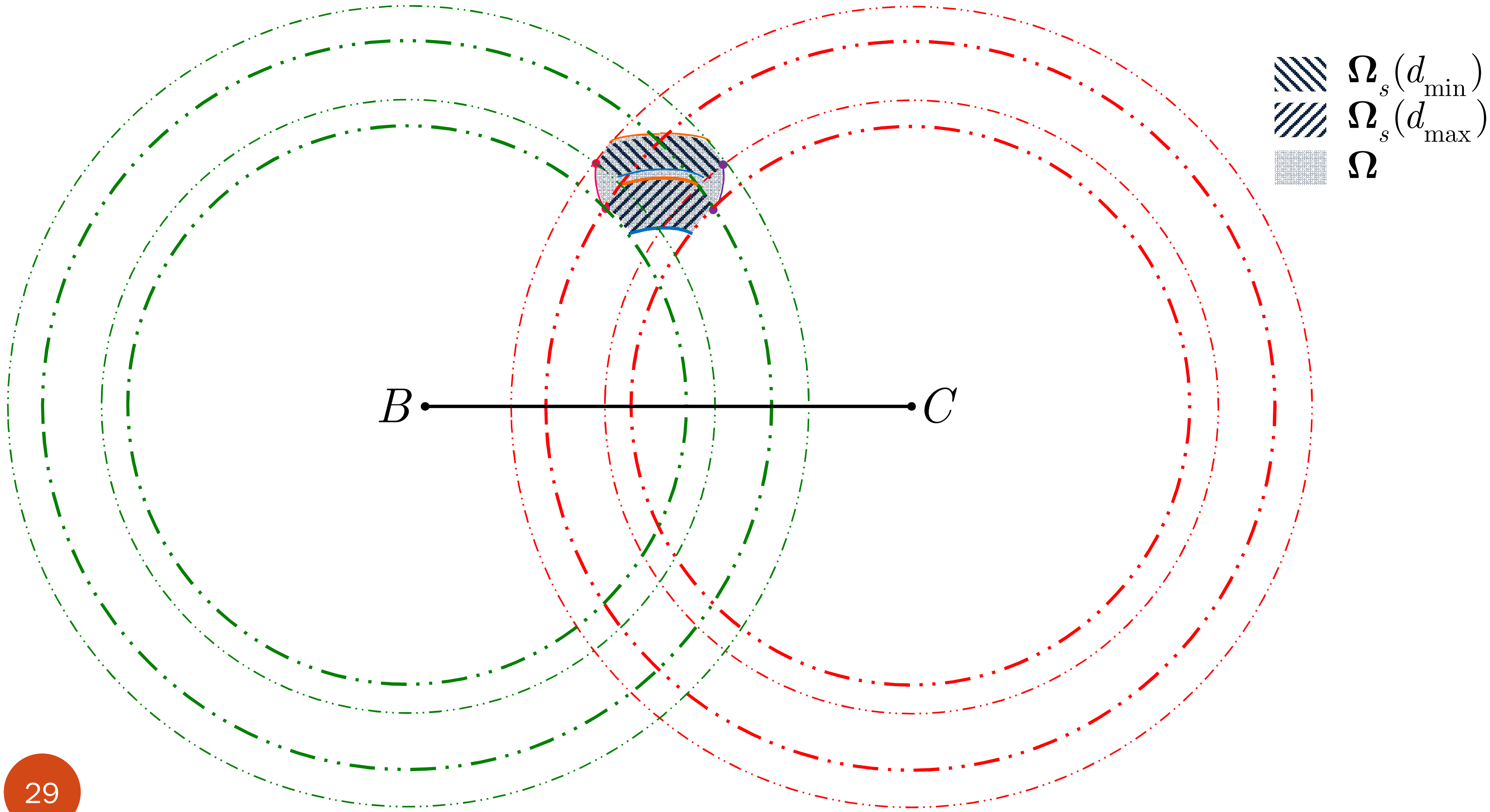
Problem Formulation

- Construction of the *joint similar feasible top vertex set* Ω

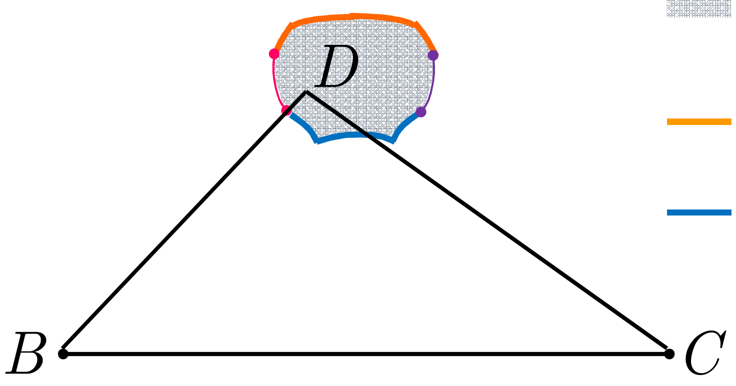


Problem Formulation

- Construction of the *joint similar feasible top vertex set* $\mathbf{\Omega}$



Problem Formulation



$\Omega = \bigcup_{d_{\min} \le d \le d_{\max}} \Omega_s(d)$

top boundary, $\partial\Omega_t$

bottom boundary, $\partial\Omega_b$

- Conformal property of similarity guarantees

$$\{\measuredangle B_d D_d C_d \mid D_d \in \mathbf{\Omega}(d),\ d_{\min} \le d \le d_{\max}\} = \{\measuredangle BDC \mid D \in \mathbf{\Omega}\}$$

- It follows

$$\boxed{\alpha_{\max} = \max_{D \in \mathbf{\Omega}} \measuredangle BDC \text{ and } \alpha_{\min} = \min_{D \in \mathbf{\Omega}} \measuredangle BDC}$$

- Advantage: only need to search into a single diagram $\mathbf{\Omega}$ for finding $\alpha_{\max}$ and $\alpha_{\min}$

Further Characteristics of Ω

- Conceptually, $\alpha_{\max}$ and $\alpha_{\min}$ will be attained by some $D \in \partial\Omega$, the boundary of Ω
- Indeed, the candidate region with regard to the identification of $\alpha_{\max}$ and $\alpha_{\min}$ can be narrowed down to $\partial\Omega$
- By means of plane geometry analyses, we have the following key theorem

Theorem 1:

(1) $\alpha_{\max} = \measuredangle BDC$ for some $D \in \partial\Omega_b$, the bottom boundary of Ω

(2) $\alpha_{\min} = \measuredangle BD'C$ for some $D' \in \partial\Omega_t$, the top boundary of Ω

Main Results

- By conducting plane geometry analyses, closed-form formula for $\alpha_{\max}$ can be obtained

Theorem 2 **:** Let $d_{\max}$ and $\tilde{d}_{\min}$ be defined as above.

- Assume that $\tilde{d}_{\min}^2 - d_{\max}^2 \geq 0$. Then we have

$$\alpha_{\max} = \cos^{-1}\left(\frac{\tilde{d}_{\min}^2 - d_{\max}^2}{\tilde{d}_{\min}^2 + d_{\max}^2}\right)$$

- Assume that $\tilde{d}_{\min}^2 - d_{\max}^2 < 0$. The following results hold

(1) if $4 \geq (\tilde{d}_{\min}^2 + d_{\max}^2)$, then

$$\alpha_{\max} = \cos^{-1}\left\{\max\left\{-1, \left(\frac{\tilde{d}_{\min}^2 - d_{\max}^2}{4\sqrt{1-\delta}\sqrt{(\tilde{d}_{\min}^2 + d_{\max}^2)/2 - (1-\delta)}}\right)\right\}\right\}$$

(2) if $4 < (\tilde{d}_{\min}^2 + d_{\max}^2)$, then

$$\alpha_{\max} = \cos^{-1}\left\{\max\left\{-1, \left(\frac{\tilde{d}_{\min}^2 - d_{\max}^2}{4\sqrt{(\tilde{d}_{\min}^2 + d_{\max}^2)/2 - (1+\delta)}\sqrt{1+\delta}}\right)\right\}\right\}$$

Main Results

- Similarly, closed-form formula for $\alpha_{\min}$ can be derived as follows

Theorem 3 : Let $d_{\max}$ and $\tilde{d}_{\min}$ be defined as above.

- Assume that $\tilde{d}_{\max}^2 - d_{\min}^2 < 0$. Then we have

$$\alpha_{\min} = \cos^{-1}\left(\frac{\tilde{d}_{\max}^2 - d_{\min}^2}{\tilde{d}_{\max}^2 + d_{\min}^2}\right)$$

- Assume that $\tilde{d}_{\max}^2 - d_{\min}^2 \geq 0$. The following results hold

(1) if $4 > (\tilde{d}_{\max}^2 + d_{\min}^2)$, then

$$\alpha_{\min} = \cos^{-1}\left(\min\left\{1, \cos\alpha_1\right\}\right)$$

where $$\cos\alpha_1 = \frac{(\tilde{d}_{\max}^2 - d_{\min}^2)}{4\sqrt{1-\delta}\sqrt{(\tilde{d}_{\max}^2 + d_{\min}^2)/2 - (1-\delta)}}$$

Main Results

(2) if $4 \le (\tilde{d}_{\max}^2 + d_{\min}^2)$, then

(a) if $(1+\delta) - d_{\min}^2 \ge 0$, then

$$\alpha_{\min} = \cos^{-1}\left(\min\{1, \cos\alpha_2\}\right)$$

where

$$\cos\alpha_2 = \begin{cases} \dfrac{1}{4\sqrt{1+\delta}} \max\left[\dfrac{(\tilde{d}_{\max}^2 - d_{\min}^2)}{\sqrt{(\tilde{d}_{\max}^2 + d_{\min}^2)/2 - (1+\delta)}}, \dfrac{(\tilde{d}_{\min}^2 - d_{\min}^2)}{\sqrt{(\tilde{d}_{\min}^2 + d_{\min}^2)/2 - (1+\delta)}}\right] & \text{, if } 4 < (\tilde{d}_{\min}^2 + d_{\min}^2); \\ \max\left\{\dfrac{(\tilde{d}_{\max}^2 - d_{\min}^2)}{4\sqrt{1+\delta}\sqrt{(\tilde{d}_{\max}^2 + d_{\min}^2)/2 - (1+\delta)}}, \dfrac{(1+\delta) + (1-\delta) - d_{\min}^2}{2\sqrt{1+\delta}\sqrt{1-\delta}}\right\} & \text{, if } 4 \ge (\tilde{d}_{\min}^2 + d_{\min}^2). \end{cases}$$

(b) if $(1+\delta) - d_{\min}^2 < 0$, then

$$\alpha_{\min} = \cos^{-1}\left(\min\{1, \cos\alpha_3\}\right)$$

where $$\cos\alpha_3 = \frac{(\tilde{d}_{\max}^2 - d_{\min}^2)}{4\sqrt{1+\delta}\sqrt{(\tilde{d}_{\max}^2 + d_{\min}^2)/2 - (1+\delta)}}$$

Discussions

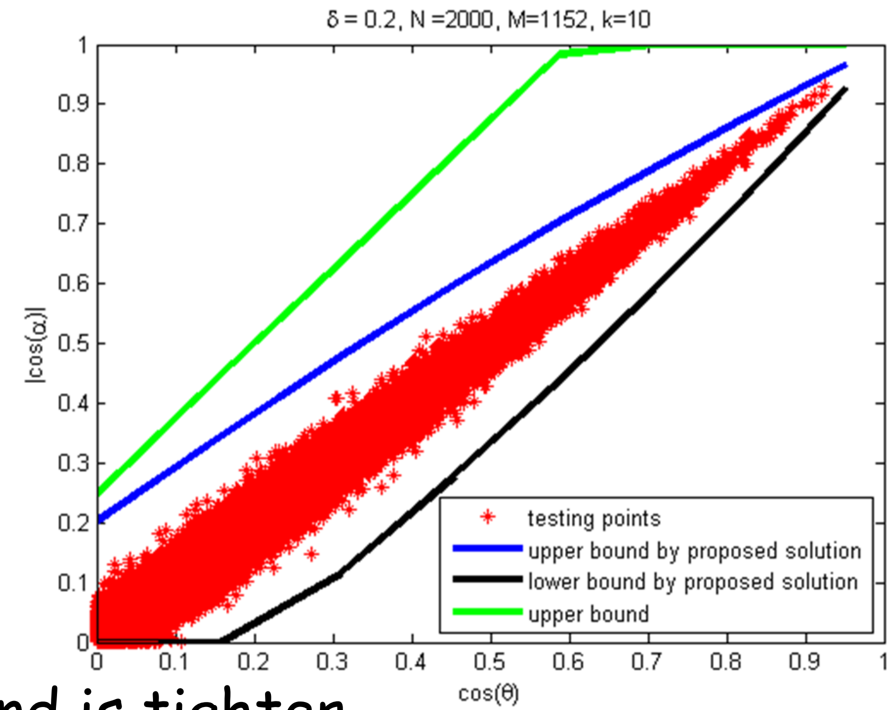
- Upper bounds of $|\cos\measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v})|$ are central to performance evaluations of many CS systems
  - Existing solution based on polarization identity

$$|\cos\measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v})| \le \min\left\{\frac{\delta + 2|\cos\theta|}{2(1-\delta)},\ 1\right\}$$

- Given $\alpha_{\max}$ and $\alpha_{\min}$ derived above, the achievable $|\cos\measuredangle(\mathbf{\Phi u}, \mathbf{\Phi v})| = |\cos\alpha|$ can be specified as

$$\begin{cases} |\cos(\alpha_{\max})| \le |\cos(\alpha)| \le |\cos(\alpha_{\min})|, & \text{if } 0 < \alpha_{\min} < \alpha_{\max} \le \pi/2, \\ |\cos(\alpha_{\min})| \le |\cos(\alpha)| \le |\cos(\alpha_{\max})|, & \text{if } \pi/2 < \alpha_{\min} < \alpha_{\max} < \pi, \\ 0 \le |\cos(\alpha)| \le \max\{|\cos(\alpha_{\min})|, |\cos(\alpha_{\max})|\}, & \text{if } 0 < \alpha_{\min} < \pi/2 < \alpha_{\max} < \pi. \end{cases}$$

Discussions



- Proposed upper bound is tighter
- The derived $\alpha_{\max}$ and $\alpha_{\min}$ are evidenced through computer simulations
  - $\mathbf{\Phi} \in \mathbb{R}^{m \times p}$, $\mathbf{\Phi}_{ij} \sim \mathcal{N}(0,1)$
  - RIC of $\mathbf{\Phi}$, $\delta = 0.2$
  - Sparsity level: $k = 15$
  - $m = 1152, p = 2000$
  - T=20000 sparse vector pairs $\{(\mathbf{u}_i, \mathbf{v}_i) \,|\, 1 \leq i \leq T\}$ are generated
  - $\theta_i = \measuredangle(\mathbf{u}_i, \mathbf{v}_i)$ and $|\cos \alpha_i|$ are computed and plotted

Application[7]

- Interference cancellation in the framework of CS[1][4]
- Data model $\mathbf{y} = \mathbf{\Phi}(\mathbf{u} + \mathbf{v})$
  - $\mathbf{u}$: desired sparse signal with support T_u
  - $\mathbf{v}$: sparse interference with support T_v
  - $T_u \cap T_v$ is empty, thus $\theta = \measuredangle(\mathbf{u}, \mathbf{v}) = \pi / 2$
  - T_v is known
- Interference can be removed by projecting $\mathbf{y}$ onto orthogonal complement of interference subspace

$$\mathbf{P}\mathbf{y} = \mathbf{P}\mathbf{\Phi}(\mathbf{u} + \mathbf{v}) = \mathbf{P}\mathbf{\Phi}\mathbf{u}$$

  - $\mathbf{P}$ is a certain orthogonal projection matrix
  - $\mathbf{P}\mathbf{\Phi}$ is the effective sensing matrix

[7] L. H. Chang and J. Y. Wu, "Compressive-domain interference cancellation via orthogonal projection: How small the restricted isometry constant of the effective sensing matrix can be?" *Proc. IEEE Wireless Communications and Networking Conference (IEEE WCNC), 2012*, pp. 256-261, Paris, France.

Application

- The RIC of $\mathbf{P\Phi}$, denoted by $\tilde{\delta}$, is crucial for system performance evaluation

$$(1-\tilde{\delta})\|\mathbf{u}\|_2^2 \le \|\mathbf{P\Phi u}\|_2^2 \le (1+\tilde{\delta})\|\mathbf{u}\|_2^2$$

✓ Smaller $\tilde{\delta}$ → Improved robustness of signal recovery

✓ FACT [4]: Suppose $\mathbf{P\Phi}$ satisfies RIP of order $2|T_u|$ with $\tilde{\delta} < \sqrt{2}-1$. Given measurements of the form $\mathbf{y} = \mathbf{P\Phi u} + \mathbf{e}$ where $\|\mathbf{e}\|_2 \le \varepsilon$, the solution $\hat{\mathbf{u}}$ to

$$\boxed{\text{Minimize } \|\mathbf{u}\|_1 \text{, subject to } \|\mathbf{y}-\mathbf{P\Phi u}\|_2 \le \varepsilon}$$

satisfies $\|\hat{\mathbf{u}}-\mathbf{u}\|_2 \le \dfrac{4(1+\tilde{\delta})}{1-(\sqrt{2}+1)\tilde{\delta}}\varepsilon$

✓ $\dfrac{4(1+\tilde{\delta})}{1-(\sqrt{2}+1)\tilde{\delta}}$ is small if $\tilde{\delta}$ is small

Application

- An upper bound of the RIC $\tilde{\delta}$ proposed in [1], [4] is

$$\tilde{\delta} \le \frac{\delta}{1-\delta}$$

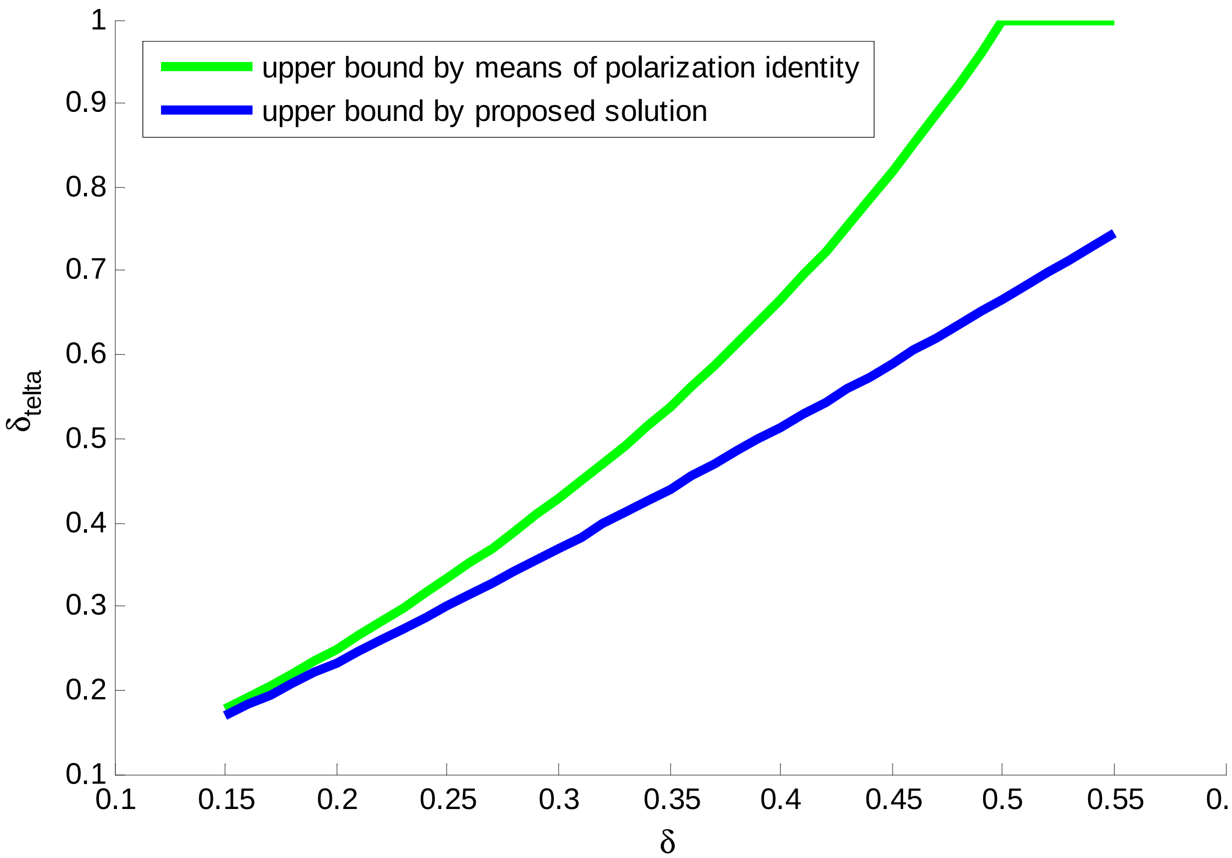
  - Derivations are based on the algebraic approach

- With the aid of $\alpha_{\max}$ and $\alpha_{\min}$, the proposed upper bound of $\tilde{\delta}$ is obtained as the following

$$\tilde{\delta} \le \begin{cases} \delta + (1-\delta) \times |\cos(\alpha_{\min})|^2, & \text{if } 0 < \alpha_{\min} < \alpha_{\max} \le \pi/2, \\ \delta + (1-\delta) \times |\cos(\alpha_{\max})|^2, & \text{if } \pi/2 < \alpha_{\min} < \alpha_{\max} < \pi, \\ \delta + (1-\delta) \times \max\left\{|\cos(\alpha_{\min})|^2, |\cos(\alpha_{\max})|^2\right\}, & \text{otherwise.} \end{cases}$$

Application

- Proposed solution v.s. algebraic based solution



- Proposed estimate is uniformly smaller
- More accurate evaluation of performance

Conclusions

- Under RIP-induced norm/distance constraints, the maximal and minimal angles between two compressed sparse vectors are analytically characterized
- Important aspects of the proposed plane-geometry based formulation
  - Law of Cosines
    - Make visualization of all algebraic RIP norm/distance constraints possible on 2-D plane
    - Auxiliary triangles depict compressed vector pairs under RIP
  - Similarity
    - Enable us to plot all auxiliary triangles in a single diagram
    - Facilitate a unified and tractable plane geometry analysis

Conclusions

- By means of plane-geometric analyses,
  - The candidate region can be narrowed down to the boundary of the constructed single diagram
  - Closed-form formulae of $\alpha_{\max}$ and $\alpha_{\min}$ can be derived
    - Verified through computer simulations
- Proposed geometric based approach v.s. existing algebraic based method
  - Algebraic method: "norm/distance inequalities + polarization identity" leads to "worst-case" estimates!
  - Proposed method:
    - All relevant algebraic constraints are jointly elucidated from the plane geometry perspective
    - Joint analyses yield tighter solutions

Conclusions

- Applications of our study to compressed-domain interference cancellation are discussed
  - Tighter estimate of the RIC of the effective sensing matrix
- Our study provides better understanding of compressed space geometry
  - In addition to already-known norm/distance metric under RIP, angle now can explicitly come into play!
  - More accurate performance evaluations of many sparse signal recovery algorithms, e.g., OMP, SP, etc. [9]

[8] L. H. Chang and J. Y. Wu, "Achievable angles between two compressed sparse vectors under norm/distance constraints imposed by the restricted isometry property: A plane geometry approach," *IEEE Trans. Information Theory*, vol. 59, no. 4, pp. 2059-2081, April 2013.

[9] L. H. Chang and J. Y. Wu, "An improved RIP-based performance guarantee for sparse signal recovery via orthogonal matching pursuit," *IEEE Trans. Information Theory*, accepted and to appear, 2014.